Effect of Mechanical Load and Thickness Profile on Creep in a Rotating Disc by using Seth's Transition Theory

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Abstract. Rotating disc is a common component in turbines, rotors, compressors and other engineering components. In this paper efforts have been made to evaluate creep stresses and strain rate in a rotating disc with respect to changes in mechanical load and thickness profile. Seth's theory of transition has been used in this study. It has been observed that stresses increases with increase in mechanical load and maximum value of strain rate further increases at the internal surface for compressible materials. It is concluded that, rotating disc is likely to fracture by cleavage close to the shaft at the bore.

Keywords: Strain, Stresses, Disc, Load, thickness, material.

INTRODUCTION

Rotating disc plays an important role in machine design. Stress analysis of rotating discs has an important role in engineering design. Rotating discs are the most critical part of rotors, turbine motors, compressor, flywheels, gears, shrink fits etc. Various machines such as centrifuges and separators for division or filtration of suspensions and emulsions, gas and steam turbine engines, turbochargers, etc. incorporate in their construction the rotating parts of various shapes, which operate under inertial (centrifugal) force and surface loads. Problem of stress calculation in a rotating thin disk was considered for the first time in [1] report "On the equilibrium of elastic solids" The classical theories of creep start with the assumptions of constitutive equations for creep and the classical theories of plasticity need an extra relation called the yield condition in addition to the flow rules. The description of the deformations in a solid subjected to external forces is thus given by a different set of equations for elastic, plastic and creep deformations. The use of rotating disc in machinery and structural applications has generated considerable interest in many problems in the domain of solid mechanics. Solutions for thin isotropic discs can be found in most of the standard creep textbooks [2 -4]. Wahl [7] has investigated creep deformation in rotating discs assuming small deformation, incompressibility condition, Tresca yield criterion, its associated flow rule and a power strain law. Mendelson [6] used an iterative scheme to obtain the thermo-plastic solution for rotating disks based on the Lame's solution. Gamer [7-8] presented more modern treatment for an elastic plastic annular disk with linear hardening behavior subjected to external pressure. You et al. [9, 10] have studied rotating disks of variable thickness and density, using a unified numerical method based on a polynomial plastic stress- strain relation and Von-Mises's yield criterion. Hojjati and Jafari [11-13] studied the elastic and elastic-plastic analyses of non-uniform thickness and density rotating disk under only centrifugal body loadings by using three semi-exact methods namely the variational iteration method, homotopy perturbation method, and adomian's decomposition method. Gupta et. al. [14] analyzed creep transition in a thin rotating disc with rigid Inclusion by using Seth transition theory. Thakur *et al.* [17] investigated thermal creep stresses and strain rates in a circular disc with a shaft having variable density by using Seth theory. Seth's transition theory does not acquire any assumptions like a yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the field and has been successfully applied to a large number of problems [14 - 19].The thickness is assumed to vary along the radius in the form:

$$h = h_0 \left(r / b \right)^{-k} \quad (1)$$

where h_0 are the thickness at $r = b$ and k (>0) are thickness parameters respectively.

GOVERNING EQUATIONS

We consider a thin circular disc with the internal radius a and external radius b respectively. The disc, produced by the material having variable thickness, is mounted on a rigid shaft. The disc is rotating with angular speed ω about a central axis perpendicular to its plane. The density of the disc is assumed to be constant and is taken sufficiently small so that the disc is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero. The displacement components in cylindrical polar coordinate are given by Seth [15, 16]:

$$u = r(1-\beta) \,, v = 0, \; w = dz \quad (2)$$

where u, v, w (displacement components); β is position function, depending on $r = \sqrt{x^2 + y^2}$ only, and d is a constant. The generalized components of strain are given by Seth [16]:

$$e_{rr} = \frac{1}{n}\left[1 - \left(r\beta' + \beta\right)^n\right], \; e_{\theta\theta} = \frac{1}{n}\left[1 - \beta^n\right], e_{zz} = \frac{1}{n}\left[1 - (1-d)^n\right], \; e_{r\theta} = e_{\theta z} = e_{zr} = 0 \quad (3)$$

where r, θ, z be polar coordinates and $\beta' = d\beta / dr$.

The stress-strain relations for isotropic material are given by [20]:

$$\sigma_{ij} = \lambda \delta_{ij} I_1 + 2\mu e_{ij} \quad (i, j = 1, 2, 3) \quad (4)$$

where σ_{ij} and e_{ij} are the stress, strain components, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta. Substituting equation (3) in equation (4), the stresses are obtained as:

$$\sigma_{rr} = \left(2\mu / n\right)\left[3 - 2C - \beta^n \left\{(1-C) + (2-C)(P+1)^n \right\}\right],$$

$$\sigma_{\theta\theta} = \left(2\mu / n\right)\frac{2\mu}{n}\left[3 - 2C - \beta^n \left\{(2-C) + (1-C)(P+1)^n \right\}\right],$$

$$\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{zr} = \sigma_{zz} = 0 \quad (5)$$

where c is the compressibility factor of the material in term of Lame's constant, and are given by $C = 2\mu / \lambda + 2\mu$.

The equation of equilibrium is:

$$\frac{d}{dr}\left(hr\sigma_{rr}\right) - h\sigma_{\theta\theta} + \rho\omega^2 r^2 h = 0 \quad (6)$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial and circumferential stresses, ρ is the material density of the rotating disc and ω is the constant angular speed.

Using equation (5) and equation (1) in equation (6), we get a non- linear differential equation in β as:

$$(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta} = \left[\left(\frac{rh'}{h}\right)\left[3 - 2C - \beta^n \begin{Bmatrix} 1-C \\ +(2-C)(P+1)^n \end{Bmatrix}\right] + \beta^n \begin{Bmatrix} 1-(P+1)^n \\ -nP\left[1-C+(2-C)(P+1)^n\right] \end{Bmatrix} + \frac{n\rho\omega^2 r^2}{2\mu}\right] \quad (7)$$

where $r\beta' = \beta P$ (P is a function of β and β is the function of r). Transition points of β in equation (7) are $P \to -1, \; \pm\infty$.

Boundary conditions: The boundary condition are:

$$\sigma_{rr} = 0 \text{ at } r = a \text{ and } \sigma_{rr} = L_0 \text{ at } r = b \quad (8)$$

where L_0 load applied at the external surface of the disc.

SOLUTION THROUGH THE PRINCIPAL STRESS DIFFERENCE

For finding the thermal creep stresses and strain rates, the transition function is taken through principal stress difference (see Seth [15, 16]; Gupta *et al.* [14] and Thakur *et al.* [17 - 19]) at the transition point $P \to -1$. We define the transition function T_f as:

$$T_f = \sigma_{rr} - \sigma_{\theta\theta} = \frac{2\mu\beta^n}{n}\left[1-(P+1)^n\right] \tag{9}$$

where T_f is a function of *r* only and T_f is dimension.

Taking the logarithmic differentiation of equation (9) with respect to *r* and substituting equation (7) and taking asymptotic value $P \to -1$, we get:

$$\frac{d}{dr}(\ln T_f) = -\left[\frac{n(3-2C)+1}{r(2-C)}\right] + \frac{h'}{h}\left(\frac{1-C}{2-C}\right) - \frac{r^n}{D^n(2-C)}\left[\frac{h'}{h}(3-2C) + \frac{n\rho\omega^2 r^2}{2\mu}\right] \tag{10}$$

The asymptotic value of β as $P \to -1$ is *D/r*; *D* being a constant. Integrating equation (10) with respect to *r*, we get

$$T_f = \sigma_{rr} - \sigma_{\theta\theta} = A r^d h^\nu \exp f, \tag{11}$$

where A is a constant of integration ,which can be determined by boundary condition and $\nu = \frac{1-C}{2-C}$ (Poisson ratios), $d = -\left[\frac{n(3-2C)+1}{(2-C)}\right]$ and $f = -\frac{1}{(2-C)D^n}\int\left[(3-2C)\frac{h'}{h} + \frac{n\rho\omega^2 r}{2\mu}\right] r^n dr$. From equations (11) and (6), we have

$$h\sigma_{rr} = B - A\int F dr - \rho\omega^2 \int rh dr \tag{12}$$

where *B* is a constant of integration, which can be determined by boundary condition and $F = r^{d-1} h^{\nu+1} \exp f$.

Using boundary condition of equation (8) in equation (12), we get $A = -\left[\rho\omega^2 \int_a^b rh dr + h_0 T_0\right] / \int_a^b F dr$;

$B = \rho\omega^2 \int_{r=a} h dr + A \int_{r=a} F dr$. Substituting the value of constant *A* and *B* in equation (12), we get:

$$\sigma_{rr} = \left[\frac{\rho\omega^2 \int_a^b h r dr + h_0 T_0}{\int_a^b F dr}\right] \int_a^r F dr - \frac{\rho\omega^2}{h} \int_a^r h r dr \tag{13}$$

Substituting eq. (13) in eq. (11), we get:

$$\sigma_{\theta\theta} = \sigma_{rr} + \frac{\rho\omega^2 \int_a^b h r dr + h_0 T_0}{\int_a^b F dr} \cdot \left(\frac{rF}{h}\right) \tag{14}$$

Eqs. (13) and (14) give creep stresses in rotating disc with a mechanical load having the variable thickness. We introduce the following non-dimensional components: $R = r/b$, $R_0 = a/b$, $\sigma_r = \sigma_{rr}/L_0$, $\sigma_\theta = \sigma_{\theta\theta}/L_0$, $E_1 = E/L_0$ and $\Omega^2 = \rho\omega^2 b^2 / L_0$. Equations (13) to (14) in non-dimensional form become:

$$\sigma_r = \frac{\left[\Omega^2\left(\frac{1-R_0^{2-k}}{2-k}\right)+1\right]}{\int_{R_0}^{1} F_1 dR} R^k \int_{R_0}^{R} F_1 dR - \frac{\Omega^2 R^k \left(R^{2-k} - R_0^{2-k}\right)}{(2-k)} \tag{15}$$

$$\sigma_\theta = \sigma_r + \frac{\left[\Omega^2\left(\frac{1-R_0^{2-k}}{2-k}\right)+1\right]}{\int_{R_0}^{1} F_1 dR} R^k R^{d-k(\nu+1)} b^d \exp f_1 \tag{16}$$

where $f_1 = \frac{kb^n(3-2C)R^n}{n(2-C)D^n} - \frac{n(3-2C)\Omega^2 b^n R^{2+n}}{E_1(2-C)^2 D^n (2+n)}$ and $F_1 = R^{d-1-k(\nu+1)} b^{d-1} \exp f_1$.

Fully plastic state: For a disc made of an incompressible material ($\nu \to 1/2$ $C = 0$), equation (15) and equation (16) become:

$$\sigma_r = \frac{\left[\Omega^2\left(\frac{1-R_0^{2-k}}{2-k}\right)+1\right]}{\int_{R_0}^{1} F_2 dR} R^k \int_{R_0}^{R} F_2 dR - \frac{\Omega^2 R^k \left(R^{2-k} - R_0^{2-k}\right)}{(2-k)} \tag{17}$$

$$\sigma_\theta = \sigma_r + \frac{\left[\Omega^2\left(\frac{1-R_0^{2-k}}{2-k}\right)+1\right]}{\int_{R_0}^{1} F_2 dR} R^k R^{d-k(\nu+1)} b^d \exp f_1 \tag{18}$$

where $f_2 = \frac{3kb^n R^n}{2nD^n} - \frac{3n\Omega^2 b^n R^{2+n}}{4E_1 D^n (2+n)}$; $F_2 = R^{-1.5(n+k+1)} b^{-1.5(n+1)} \exp f_2$ (Constants); σ_r (radial stress); σ_θ (circumferential stresses), $R = r/b$ and $R_0 = a/b$ (Radii ratios).

STRAIN RATES ANALYSIS

When the creep sets in, the strains are be replaced by strain rates and the stress-strain relations in equation (4) become:

$$\dot{e}_{ij} = (3/2)\lambda_1 S_{ij} \tag{19}$$

where $\dot{e}_{ij}$ is the strain rate tensor with respect to flow parameter t and S_{ij} is the stress deviator tensor. Differentiating equation (3) with respect to time *t*, we get:

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\dot{\beta} \tag{20}$$

For SWAINGER measure (*i.e.* n = 1), equation (19) become: $\dot{\varepsilon}_{\theta\theta} = \dot{\beta}$. (21)

where $\dot{\varepsilon}_{\theta\theta}$ is the SWAINGER strain measure. From equation (9) the transition value β is

$$\beta = (n/2\mu)^{1/n}\left[\sigma_{rr} - \sigma_{\theta\theta}\right]^{1/n} \tag{22}$$

Using equations (20), (21) and (22) in equation (19), we get:

$$\dot{\varepsilon}_{rr} = \left[n(\sigma_r - \sigma_\theta)(1+\nu)\right]^{\frac{1}{n}-1}\left[\sigma_r - \nu\sigma_\theta\right]$$
$$\dot{\varepsilon}_{\theta\theta} = \left[n(\sigma_r - \sigma_\theta)(1+\nu)\right]^{\frac{1}{n}-1}\left[\sigma_\theta - \nu\sigma_r\right]$$
$$\dot{\varepsilon}_{zz} = -\left[n(\sigma_r - \sigma_\theta)(1+\nu)\right]^{\frac{1}{n}-1}\left[\nu(\sigma_r + \sigma_\theta)\right] \quad (23)$$

where $\dot{\varepsilon}_{rr}$, $\dot{\varepsilon}_{\theta\theta}$, $\dot{\varepsilon}_{zz}$ are strain rates tensor.

These are the constitutive equations used by Odquist [21] for finding the creep stresses and strain rates provided we put $n = 1/N$.

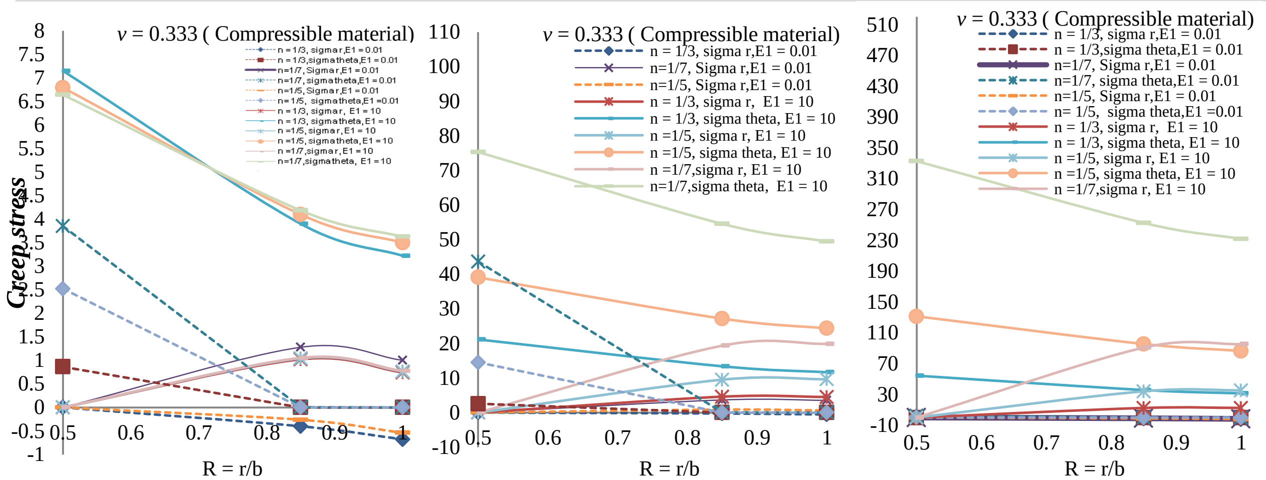
NUMERICAL ILLUSTRATION AND DISCUSSION

For calculating stresses, strain-rates based on the above analysis, the following numerical values have been taken $\Omega^2 = 2, 5$, $\nu = 0.5$ and 0.333, n = 1/3, 1/5, 1/7 (*i.e* N =3, 5, 7), k = 0, 0.25, $E_1 = 0.01, 10$ and D = 1. In classical theory measure, N is equal to $1/n$. Definite integrals in the equations (15) and (16) have been solved by using Simpson's 1/3rd rule. From Figure 1(a)-1(b), curves are produced between creep stresses along the radii ratio for the rotating disc made of compressible as well as incompressible materials with thickness k = 0, 0.25, 0.35 at different angular speed $\Omega^2 = 2,5$. It has been seen that circumferential stress are maximum at the internal surface for measure n = 1/3 (flat disc; k = 0) and n = 1/7 (k = 0.25 and 0.25) at load $E = 10$ and for measure n = 1/7 (k = 0, 0.25 and 0.35 at load $E = 0.01$. It has been observed that mechanical load increases the values of stresses. Disc made of the compressible material required the maximum value of circumferential stress as compared to one made of incompressible material (*i.e.* rubber). Angular speed increased values of circumferential stress for compressible as well as incompressible material. Curve are produced for strain rates along the radii ratio $R = r/b$ (*see* Figure 2) for rotating disc made of compressible materials as well as incompressible material with thickness k = 0 and 0.25 at different angular speed Ω^2 =5 for measure n = 1/7, 1/5, and 1/3 (*i.e.* N =7, 5, 3). It has been seen that rotating disc made of incompressible materials has maximum value of strain at the internal surface as compared to disc made of compressible material for measure n = 1/7 and n =1. With introduction of mechanical load, the maximum value of strain rates further increases at the internal surface for compressible materials.

Meaning of Sigma r = σ_r ; Sigma theta = σ_θ , E_1 (Load applied)

$$\Omega^2 = 2$$

k = 0	k = 0.25	k = 0.35



k = 0	k = 0.25	k = 0.35

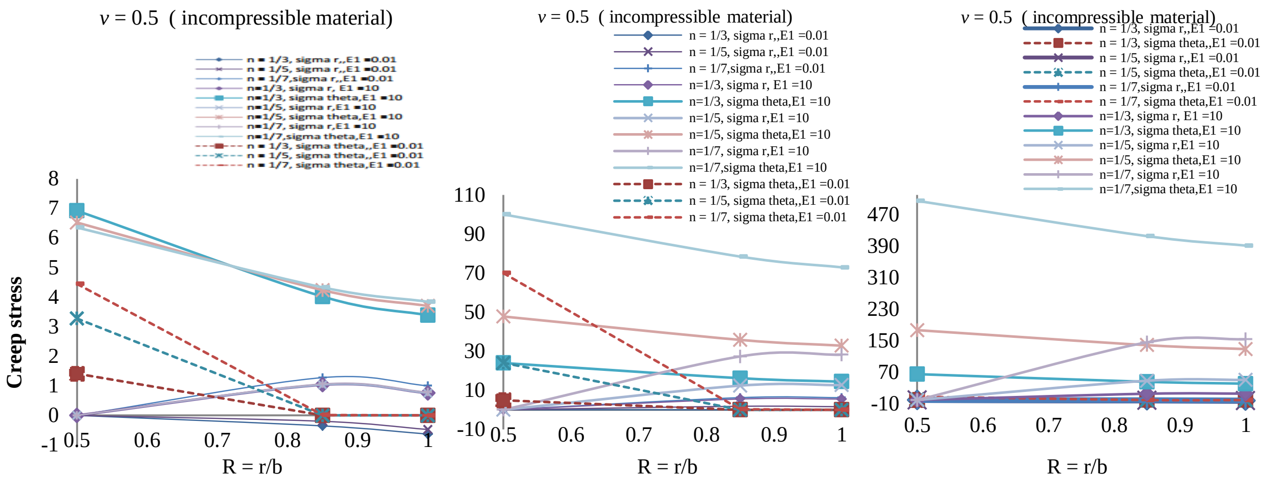


Figure. 1(a) Stresses in a thin rotating disc having variable thickness k = 0, 0.25, 0.35 and load $E_1 = 0.01, 10$ for compressible and incompressible materials.

$\Omega^2 = 5$

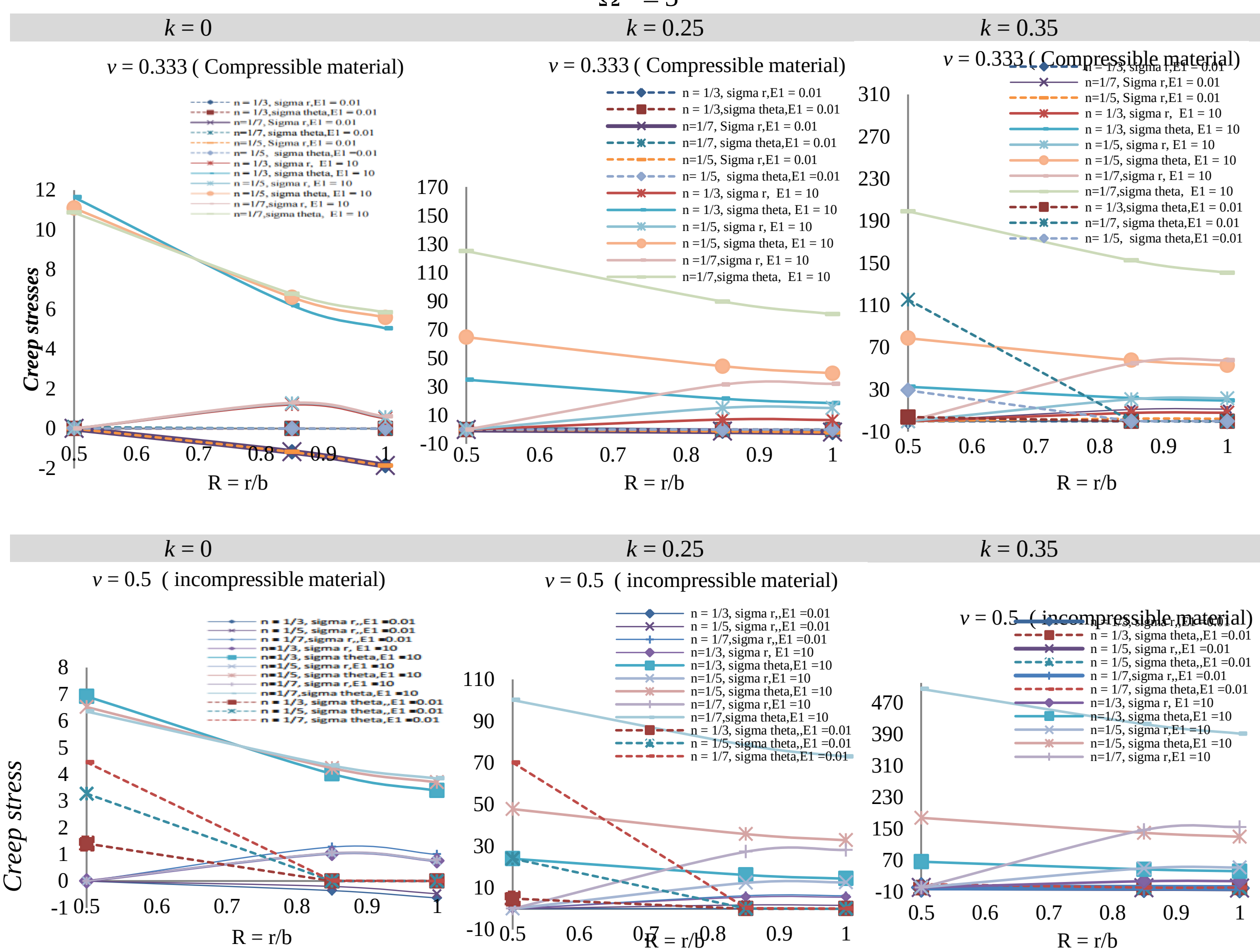


Figure. 1(b) Stresses in a thin rotating disc having variable thickness k = 0, 0.25, 0.35 and load $E_1 = 0.01, 10$ for compressible and incompressible materials.

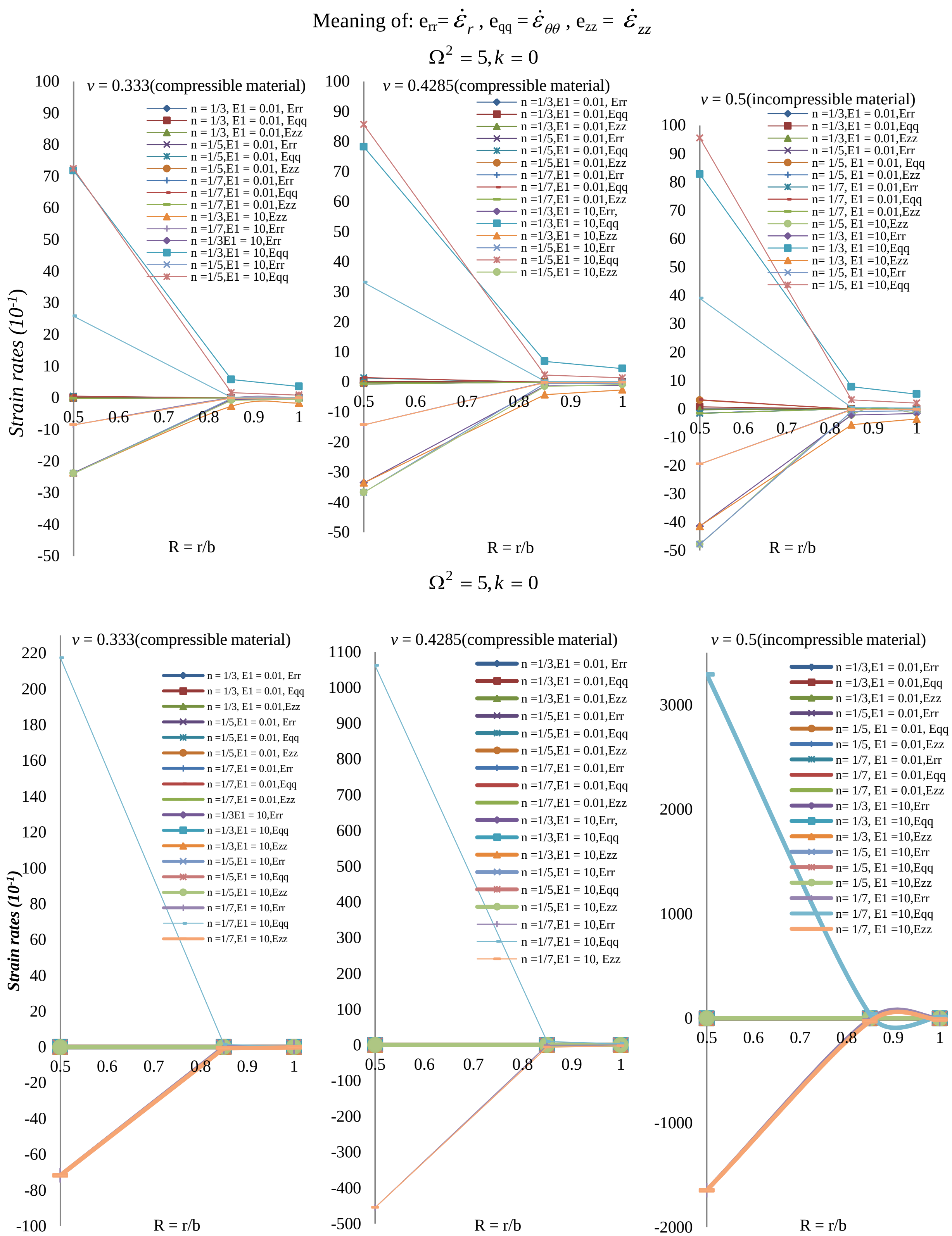


Figure 2. Creep strain rate distribution in a thin disc with thickness k = 0, 0.25 for measure n = 1/7, 1/5 , 1/3 along the radius R = r/b.

CONCLUSION

Stresses and strain rates have been obtained in a rotating disc having variable thickness parameter and subjected to mechanical load using Seth's transition theory. It has been seen that mechanical load increases the values of stresses. With the introduction of mechanical load the maximum value of strain rates further increases at the internal surface for compressible materials.

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